MATH MILESTONE # B5

OPERATIONS WITH FRACTIONS

The word, **milestone,** means “a point at which a significant change occurs.” A Math Milestone refers to a significant point in the understanding of mathematics.

To reach this milestone one should be able to compute comfortably with fractions.

Index Page

Please consult the Glossary supplied with this Milestone for mathematical terms. Consult a regular dictionary at www.dictionary.com for general English words that one does not understand fully.

You may start with the Diagnostic Test on the next page to assess your proficiency on this milestone. Then continue with the lessons with special attention to those, which address the weak areas.

Researched and written by Vinay Agarwala
Edited by Ivan Doskocil

DIAGNOSTIC TEST

1. What is the least common multiple of 9, 14, and 21?

2. Add $\frac{5}{6}$, $\frac{9}{14}$ and $\frac{11}{21}$.

3. John, Bill, and Mike shared a whole pizza. John ate $\frac{1}{4}$, Bill ate $\frac{1}{3}$, and Mike ate $\frac{5}{12}$ of the pizza. How much of the pizza was left?

4. Subtract $2\frac{5}{8}$ from $3\frac{7}{12}$.

5. Multiply $\frac{8}{27}$ by $\frac{15}{16}$ by first reducing the product to simplest terms during multiplication.

6. A person must pay $\frac{1}{6}$th of his income in taxes. If the annual income of Joe is $36,000, how much should he pay in taxes?

7. How many seconds are $\frac{3}{4}$ of a minute?

8. Multiply $1\frac{3}{5}$ by $1\frac{9}{16}$.

9. Divide $\frac{9}{16}$ by $\frac{3}{8}$.

10. Divide $6\frac{2}{5}$ by $2\frac{2}{15}$.

11. Express "4 inches" as a fraction of a foot.

12. Simplify $\frac{5}{6} - \frac{1}{2} + \frac{2}{3} - \frac{5}{9}$

13. Simplify $\dfrac{\frac{5}{7} - \frac{4}{21}}{\frac{11}{21} + \frac{3}{14}}$

Answer: **1.** 126 **2.** 2 **3.** None **4.** 23/24 **5.** 5/18 **6.** $6000 **7.** 45 **8.** 2 1/2 **9.** 3/2 **10.** 3 **11.** 1/3 ft **12.** 4/9 **13.** 22/31

LESSONS

Lesson B5.1 Least Common Multiple

We need to compute the least common multiple (LCM) of the denominators of "unlike" fractions to convert them to "like" fractions.

1. The product of two denominators provides a common multiple.

   A common multiple of 10 and 15 is 10x15 = 150. But 150 may not be the least common multiple of 10 and 15.

   Multiples of 10 are: 10, 20, <u>30</u> ...
   Multiples of 15 are: 15, <u>30</u> ...

   Thus, the least common multiple of 10 and 15 is **30**.

2. We obtain the least common multiple by removing the repeat occurrence of common factors. We have 5 as a factor common to both 10 and 15. We remove the repeat occurrence of 5 from the product to get the LCM.

   Product = 10 x 15 = (2 x **5**) x (3 x **5**) = 150
   Least Common Multiple = (2 x **5**) x (3 x ~~5~~) = 30

   The prime factors of the LCM contain the prime factors of the numbers.

30	=	<u>5</u> x <u>2</u> x 3	=	(5 x 2) x 3	=	**10** x 3		(30 is a multiple of 10)
30	=	<u>5</u> x 2 x <u>3</u>	=	(5 x 3) x 2	=	**15** x 2		(30 is a multiple of 15)

3. We may eliminate the repeat occurrence of a common factor from the product by applying short division to the numbers side by side as shown below.

   5 | <u>10, 15</u> (5 is a common factor)
   2, 3

   There are no other factors common to the quotients in the bottom row. We multiply the common factor to the remaining quotients to get the LCM.

   LCM = 5 x 2 x 3 = 30

   <u>Find the LCM of 42 and 63</u>.

   7 | <u>42, 63</u> (7 is a common factor)
   3 | <u>6, 9</u> (3 is a common factor)
   2, 3 (There is no other common factor)

   We multiply the common factors to the remaining quotients to get the LCM.

   LCM = 7 x 3 x 2 x 3 = 126

   126 is a common multiple of 42 and 63 because it contains the prime factors of 42 and 63 as follows.

126	=	7 x 3 x 2 x 3	=	(7 x 3 x 2) x 3	=	**42** x 3
126	=	7 x 3 x 2 x 3	=	(7 x 3 x 3) x 2	=	**63** x 2

4. For more than two numbers find all the prime factors that are common to at least two of those numbers to get the LCM.

   Find the LCM of 9, 14 and 21.

   3 | 9, 14, 21 (3 is a prime factor common to 9 and 21, bring 14 down as-is)
   7 | 3, 14, 7 (7 is a prime factor common to 14 and 7, bring 3 down as-is)
   3, 2, 1 (No prime factor is common to any two of these numbers)

   We multiply the common prime factors to the remaining quotients to get the LCM.

   LCM = 3 x 7 x 3 x 2 x 1 = 126

   The LCM should contain the prime factors of the numbers.

   126 = 3 x 7 x 3 x 2
   = (3 x 3) x 2 x 7 = **9** x 14
   = (7 x 2) x 3 x 3 = **14** x 9
   = (3 x 7) x 2 x 3 = **21** x 6

   Find the LCM of 42, 56 and 70.

   7 | 42, 56, 70 (7 is a prime factor common to all numbers)
   2 | 6, 8, 10 (2 is a prime factor common to all numbers)
   3, 4, 5 (No prime factor is common to any two numbers)

   LCM = 7 x 2 x 3 x 4 x 5 = 840

   840 = 7 x 2 x 3 x 4 x 5
   = (7 x 2 x 3) x 4 x 5 = **42** x 20
   = (7 x 2 x 4) x 3 x 5 = **56** x 15
   = (7 x 2 x 5) x 3 x 4 = **70** x 12

5. To convert "unlike" to "like" fractions, we compute the LCM of the denominators, and then use the LCM as the common denominator of the equivalent "like" fractions.

   Convert 5/6, 7/10, and 11/15 to like fractions.

   Find the LCM of "unlike' denominators 6, 10, and 15.

   3 | 6, 10, 15 (3 is a prime factor common to 6 and 15)
   5 | 2, 10, 5 (5 is a prime factor common to 10 and 5)
   2 | 2, 2, 1 (2 is a prime factor common to 2 and 2)
   1, 1, 1

   LCM = 3 x 5 x 2 = 30

   Therefore, the equivalent "like" fractions are:

   $$\frac{5}{6} = \frac{5 \times 5}{6 \times 5} = \frac{25}{30}$$

   $$\frac{7}{10} = \frac{7 \times 3}{10 \times 3} = \frac{21}{30}$$

   $$\frac{11}{15} = \frac{11 \times 2}{15 \times 2} = \frac{22}{30}$$

☺ Exercise B5.1

1. If you want to compare the following fractions, what is the least common denominator that you would use?

   (a) $\frac{3}{9}$ $\frac{5}{12}$ (b) $\frac{11}{14}$ $\frac{7}{8}$ (c) $\frac{7}{12}$ $\frac{8}{15}$ (d) $\frac{21}{65}$ $\frac{8}{25}$

2. Find the LCM (Least Common Multiple) of the following set of numbers:

   (a) 4 and 9
   (b) 6 and 9
   (c) 14 and 42
   (d) 35 and 42
   (e) 36 and 60
   (f) 4, 6 and 9
   (g) 6, 13 and 26
   (h) 6, 15 and 18
   (i) 8, 12 and 20
   (j) 9, 12 and 21
   (k) 16, 28, 35 and 63
   (l) 26, 33, 39 and 44
   (m) 24, 40, 48 and 56
   (n) 12, 18, 27, 30 and 40
   (o) 15, 35, 55, 75, and 165

3. Three men journey 14, 35, and 38 miles a day respectively. At what distance from the starting point is the nearest place at which they all put up?

4. A bag of marbles can be divided in equal shares among 2, 3, 4, 5, or 6 friends. What is the least number of marbles that the bag could contain?

5. Find the least number which, when divided by 9, 14, and 21, will always leave the same remainder 7.

6. Convert the following unlike fractions to like fractions by first finding the LCM of the unlike denominators.

   (a) $\frac{3}{5}$, $\frac{3}{10}$ (b) $\frac{5}{6}$, $\frac{8}{9}$ (c) $\frac{3}{4}$, $\frac{1}{6}$
   (d) $\frac{5}{9}$, $\frac{7}{12}$ (e) $\frac{3}{10}$, $\frac{4}{15}$ (f) $\frac{3}{8}$, $\frac{5}{12}$
   (g) $\frac{7}{15}$, $\frac{11}{25}$ (h) $\frac{1}{6}$, $\frac{1}{8}$ (i) $\frac{5}{9}$, $\frac{7}{15}$
   (j) $\frac{9}{14}$, $\frac{11}{21}$ (k) $\frac{19}{24}$, $\frac{11}{16}$ (l) $\frac{13}{20}$, $\frac{11}{15}$

Answer: **1.** (a) 36 (b) 56 (c) 60 (d) 325 **2.** (a) 36 (b) 18 (c) 42 (d) 210 (e) 180 (f) 36 (g) 78 (h) 90 (i) 120 (j) 252 (k) 1260 (l) 1716 (m) 1680 (n) 1080 (o) 5775 **3.** 1330 miles away (the LCM) **4.** 60 **5.** 133 (7 more than the LCM) **6.** (a) 6/10, 3/10 (b) 15/18, 16/18 (c) 9/12, 2/12 (d) 20/36, 21/36 (e) 9/30, 8/30 (f) 9/24, 10/24 (g) 35/75, 33/75 (h) 4/24, 3/24 (i) 25/45, 21/45 (j) 27/42, 22/42 (k) 38/48, 33/48 (l) 39/60, 44/60

Lesson B5.2 Addition and Subtraction

We can add and subtract the multiples of like unit fractions, just as we add and subtract multiples of like units.

1. Only the multiples of "like" unit fractions may be added.

   (a) To add "like" fractions, we add the numerators.

   Add $\frac{1}{4}$ and $\frac{2}{4}$

   $\frac{1}{4} + \frac{2}{4}$ = 1 if $\frac{1}{4}$ + 2 of $\frac{1}{4}$ = **3 of $\frac{1}{4}$**

   Or, $\frac{1}{4} + \frac{2}{4} = \frac{1+2}{4} = \frac{3}{4}$

Add $\frac{2}{21}$, $\frac{5}{21}$, and $\frac{8}{21}$

$$2 \text{ of } \frac{1}{21} + 5 \text{ of } \frac{1}{21} + 8 \text{ of } \frac{1}{21} = 15 \text{ of } \frac{1}{21}$$

Or, $\frac{2}{21} + \frac{5}{21} + \frac{8}{21} = \frac{2+5+8}{21} = \frac{15}{21} = \frac{5}{7}$

The resulting sum $\frac{15}{21}$ is reduced to $\frac{5}{7}$ by factoring out 3.

(b) To add "unlike" fractions, we convert them to "like" fractions first, and then add the numerators.

Add $\frac{5}{6}$ and $\frac{7}{9}$.

The LCM of the unlike denominators 6 and 9 is **18**. We create equivalent like fractions with denominator 18, and then add the numerators.

$$\frac{5}{6} + \frac{7}{9} = \frac{(5 \times 3) + (7 \times 2)}{18}$$

$$= \frac{15 + 14}{18}$$

$$= \frac{29}{18}$$

Add $\frac{5}{6}$, $\frac{9}{14}$ and $\frac{11}{21}$.

The LCM of 6, 14, and 21 is **42**. Therefore,

$$\frac{5}{6} + \frac{9}{14} + \frac{11}{21} = \frac{5 \times 7 + 9 \times 3 + 11 \times 2}{42}$$

$$= \frac{35 + 27 + 22}{42}$$

$$= \frac{84}{42} = 2$$

2. Subtraction is the inverse of addition.

(a) To find the difference between two "like" fractions, simply subtract one numerator from the other.

Subtract $\frac{3}{25}$ from $\frac{18}{25}$.

We subtract one numerator from the other.

$$\frac{18}{25} - \frac{3}{25} = \frac{18-3}{25} = \frac{15}{25} = \frac{3}{5}$$

The denominator indicates the unit fraction. It, obviously, does not change.

(b) We must convert "unlike" fractions to "like" fractions before subtracting the numerators.

Subtract $\frac{7}{9}$ from $\frac{5}{6}$.

We convert "unlike" fractions to "like" fractions first.

$$\frac{5}{6} - \frac{7}{9} = \frac{5\text{x}3 - 7\text{x}2}{18}$$

$$= \frac{15 - 14}{18}$$

$$= \frac{1}{18}$$

Subtract $\frac{7}{18}$ from $\frac{4}{15}$.

When the subtrahend is greater than the minuend, a negative sign is placed before the fraction obtained as the difference to indicate "shortage."

$$\frac{4}{15} - \frac{7}{18} = \frac{4\text{x}6 - 7\text{x}5}{90}$$

$$= \frac{24 - 35}{90}$$

$$= -\frac{11}{90}$$

NOTE: The fractional notation represents a single quantity.

3. Mixed numbers may be added or subtracted as improper fractions, or as mixed numbers.

(a) We may convert the mixed numbers to improper fractions. Then convert them to like fractions and add. Then convert the sum back to a mixed number.

Addition

$$2\frac{5}{8} + 3\frac{7}{12} = \frac{21}{8} + \frac{43}{12} \quad \text{(Convert to improper fractions)}$$

$$= \frac{63 + 86}{24}$$

$$= \frac{149}{24}$$

$$= 6\frac{5}{24} \quad \text{(Convert back to mixed number)}$$

Subtraction

$$3\frac{7}{12} - 2\frac{5}{8} = \frac{43}{12} - \frac{21}{8} \quad \text{(Convert to improper fractions)}$$

$$= \frac{86 - 63}{24}$$

$$= \frac{23}{24}$$

(b) Alternatively, we may add the integer and fraction portions separately.

Addition

$$2\frac{5}{8} + 3\frac{7}{12} = (2+3) + \left(\frac{5}{8} + \frac{7}{12}\right)$$
$$= 5 + \frac{15 + 14}{24}$$
$$= 5 + \frac{29}{24}$$
$$= 5 + 1\frac{5}{24}$$
$$= 6\frac{5}{24}$$

Subtraction

$$3\frac{7}{12} - 2\frac{5}{8} = (3-2) + \left(\frac{7}{12} - \frac{5}{8}\right)$$
$$= 1 + \frac{14 - 15}{24}$$
$$= 1 + \left(-\frac{1}{24}\right)$$
$$= 1 - \frac{1}{24}$$
$$= \frac{24}{24} - \frac{1}{24}$$
$$= \frac{23}{24}$$

☺ Exercise B5.2

1. Add and reduce the sum to simplest terms.

(a) $\frac{3}{8} + \frac{5}{8}$ (d) $\frac{2}{9} + \frac{4}{9}$ (g) $\frac{5}{12} + \frac{1}{12} + \frac{2}{12}$

(b) $\frac{3}{10} + \frac{2}{10}$ (e) $\frac{5}{14} + \frac{2}{14}$ (h) $\frac{4}{33} + \frac{13}{33} + \frac{5}{33}$

(c) $\frac{7}{18} + \frac{5}{18}$ (f) $\frac{5}{16} + \frac{3}{16}$ (i) $\frac{8}{45} + \frac{3}{45} + \frac{4}{45}$

2. Add the following. Reduce the sum to its simplest form.

(a) $\frac{3}{4} + \frac{5}{6}$ (d) $\frac{2}{9} + \frac{5}{12}$ (g) $\frac{7}{12} + \frac{8}{15} + \frac{11}{20}$

(b) $\frac{4}{5} + \frac{3}{4}$ (e) $\frac{5}{14} + \frac{3}{7}$ (h) $\frac{2}{15} + \frac{1}{6} + \frac{3}{10}$

(c) $\frac{4}{15} + \frac{7}{12}$ (f) $\frac{15}{16} + \frac{3}{24}$ (i) $\frac{7}{15} + \frac{3}{25} + \frac{11}{35}$

3. Subtract the following.

(a) $\frac{3}{4} - \frac{1}{2}$ (d) $\frac{2}{3} - \frac{1}{6}$ (g) $\frac{3}{16} - \frac{17}{40}$

(b) $\frac{7}{12} - \frac{5}{12}$ (e) $\frac{11}{18} - \frac{4}{15}$ (h) $\frac{48}{65} - \frac{19}{26}$

(c) $\frac{5}{12} - \frac{13}{14}$ (f) $\frac{21}{25} - \frac{29}{35}$ (i) $\frac{29}{55} - \frac{17}{33}$

4. Add the following. Reduce the sum to mixed numbers in simplest form.

(a) $1\frac{3}{4} + 1\frac{5}{6}$ (d) $3\frac{2}{9} + 1\frac{5}{12}$ (g) $3\frac{7}{12} + 2\frac{8}{15} + 1\frac{11}{20}$

(b) $5\frac{4}{5} + 9\frac{3}{4}$ (e) $8\frac{5}{14} + 3\frac{3}{7}$ (h) $5\frac{2}{15} + 4\frac{1}{6} + 1\frac{3}{10}$

(c) $2\frac{4}{15} + 3\frac{7}{12}$ (f) $3\frac{15}{16} + 4\frac{3}{24}$ (i) $2\frac{7}{15} + 3\frac{3}{25} + 4\frac{11}{35}$

5. Subtract the following. Reduce the sum to mixed numbers in simplest form.

(a) $1\frac{5}{6} - 1\frac{3}{4}$ (d) $3\frac{2}{9} - 1\frac{5}{12}$ (g) $3\frac{7}{12} - 2\frac{8}{15}$

(b) $9\frac{3}{4} - 5\frac{4}{5}$ (e) $8\frac{5}{14} - 3\frac{3}{7}$ (h) $5\frac{2}{15} - 4\frac{1}{6}$

(c) $3\frac{7}{12} - 2\frac{4}{15}$ (f) $4\frac{3}{24} - 3\frac{15}{16}$ (i) $4\frac{11}{35} - 3\frac{3}{25}$

6. John, Bill, and Mike shared a whole pizza. John ate $\frac{1}{4}$, Bill ate $\frac{1}{3}$, and Mike ate $\frac{5}{12}$ of the pizza. How much of the pizza was left?

7. Bill traveled $1\frac{7}{10}$ miles to school, then $2\frac{1}{2}$ miles to the grocery store, and then $1\frac{3}{5}$ miles back home. What is the total distance he traveled?

8. Bill traveled $2\frac{1}{2}$ miles east, and then returned $1\frac{7}{10}$ miles west. How far is he from the starting point?

9. Practice addition and subtraction of fractions from other math text books or by creating exercises of your own.

Answer: **1.** (a) 1 (b) 1/2 (c) 2/3 (d) 2/3 (e) 1/2 (f) 1/2 (g) 2/3 (h) 2/3 (i) 1/3 **2.** (a) 19/12 (b) 31/20 (c) 17/20 (d) 23/36 (e) 11/14 (f) 17/16 (g) 5/3 (h) 3/5 (i) 473/525 **3.** (a) 1/4 (b) 1/6 (c) – 43/84 (d) 1/2 (e) 31/90 (f) 2/175 (g) – 19/80 (h) 1/130 (i) 2/165 **4.** (a) 3 7/12 (b) 15 11/20 (c) 5 17/20 (d) 4 23/36 (e) 11 11/14 (f) 8 1/16 (g) 7 2/3 (h) 10 3/5 (i) 9 473/525 **5.** (a) 1/12 (b) 3 19/20 (c) 1 19/60 (d) 1 29/36 (e) 4 13/14 (f) 3/16 (g) 1 1/20 (h) 29/30 (i) 1 34/75 **6.** None **7.** 5 4/5 miles **8.** 4/5 miles east

Lesson B5.3 Multiplication and Division

The multiplication of fractions is similar to mixed operations treating numerators as dividends and denominators as divisors.

1. Multiplication of fractions may be converted to mixed operations involving multiplication and division with natural numbers.

3/4 x 5/8 = (3 ÷ 4) x (5 ÷ 8) = 3 ÷ 4 x 5 ÷ 8

In mixed operations that involve multiplication and division, we may divide the product of dividends by the product of divisors.

3 ÷ 4 x 5 ÷ 8 = (3 x 5) ÷ (4 x 8)

Similarly, in multiplication of fractions, we may directly "divide" the product of numerators by the product of denominators.

$$\frac{3}{4} \times \frac{5}{8} = \frac{3 \times 5}{4 \times 8} = \frac{15}{32}$$

<u>Multiply $\frac{3}{4}$ by $\frac{2}{3}$</u>

After the multiplication of fractions, one may simplify, or reduce the product to its lowest terms, as usual.

$$\frac{3}{4} \times \frac{2}{3} = \frac{3 \times 2}{4 \times 3} = \frac{6}{12} = \frac{1}{2}$$

<u>What is half of half?</u>

We already know that half of half is a quarter. "Of" translates as multiplication in mathematics. Therefore,

$$\frac{1}{2} \text{ of } \frac{1}{2} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

2. Similar to mixed operations, we may cancel out factors that are common to numerators and denominators.

<u>Multiply $\frac{8}{27}$ by $\frac{15}{16}$</u>

$$\frac{8}{27} \times \frac{15}{16} = \frac{{}^{1}\cancel{8}}{27} \times \frac{15}{\cancel{16}_{2}} \quad \text{(factor out 8)}$$

$$= \frac{1}{{}_{9}\cancel{27}} \times \frac{\cancel{15}^{5}}{2} \quad \text{(factor out 3)}$$

$$= \frac{1}{9} \times \frac{5}{2}$$

$$= \frac{5}{18}$$

<u>Multiply $\frac{3}{4}$ by $\frac{2}{3}$</u>

$$\frac{3}{4} \times \frac{2}{3} = \frac{{}^{1}\cancel{3} \times \cancel{2}^{1}}{{}_{2}\cancel{4} \times \cancel{3}_{1}} = \frac{1 \times 1}{2 \times 1} = \frac{1}{2}$$

Here we factor out 3, and then factor out 2. The process of canceling out common factors makes the multiplication easier and reduces the error in computation. It provides a product already reduced to its lowest terms.

3. To multiply a fraction by a whole number, simply use 1 for the denominator of the whole number and follow the same procedure as above.

<u>Multiply $\frac{3}{8}$ by 2</u>

$$\frac{3}{8} \times 2 = \frac{3}{\cancel{8}_4} \times \frac{\cancel{2}^1}{1} = \frac{3 \times 1}{4 \times 1} = \frac{3}{4}$$

4. If there is "no donominator," then the number is considered a "numerator."

   (a) Three fifths of a class of 35 is girls. How many girls are in that class?

$$\frac{3}{5} \text{ of } 35 = \frac{3}{5} \times 35$$
$$= \frac{3}{\cancel{5}_1} \times \frac{\cancel{35}^7}{1}$$
$$= \frac{3 \times 7}{1 \times 1}$$
$$= 21$$

   (b) A person must pay 1/6th of his income in taxes. If the annual income of Joe is $36,000, how much should he pay in taxes?

$$\text{Taxes} = \frac{1}{6} \text{ of } \$36{,}000$$
$$= \frac{1}{\cancel{6}_1} \times \overset{6000}{\cancel{36000}} = \$6000$$

   We may assume the denominator of the whole number to be 1 without writing it.

   (c) What is a hundredth of a dollar?

$$\frac{1}{100} \text{ of } \$1 = \frac{1}{100} \text{ of } 100 \text{ cents}$$
$$= \frac{1}{\cancel{100}} \times \cancel{100} \text{ cents} = 1 \text{ cent}$$

   We may convert a quantity to smaller units to avoid fractions.

5. To multiply mixed numbers, convert them to improper fractions first.

<u>What is $1\frac{1}{2}$ of $2\frac{1}{2}$?</u>

$$2\frac{1}{2} \times 1\frac{1}{2} = \frac{5}{2} \times \frac{3}{2}$$
$$= \frac{15}{4}$$
$$= 3\frac{3}{4}$$

<u>Multiply $1\frac{3}{5}$ by $1\frac{9}{16}$</u>

$$1\frac{3}{5} \times 1\frac{9}{16} = \frac{^1\cancel{8}}{_1\cancel{5}} \times \frac{\cancel{25}^5}{\cancel{16}_2}$$

$$= \frac{1}{1} \times \frac{5}{2}$$

$$= \frac{5}{2} = 2\frac{1}{2}$$

6. The reciprocal of a number is obtained by switching numerator and denominator. The product of a number and its reciprocal is always 1.

   (a) The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, and their product is 1.

   $$\frac{2}{3} \times \frac{3}{2} = \frac{{}^{1}\cancel{2}}{{}_{1}\cancel{3}} \times \frac{\cancel{3}^{1}}{\cancel{2}_{1}} = \frac{1}{1} = 1$$

   (b) The reciprocal of 2 is $\frac{1}{2}$ because the denominator of all whole numbers is 1.

   $$2 \times \frac{1}{2} = \frac{{}^{1}\cancel{2}}{1} \times \frac{1}{\cancel{2}_{1}} = \frac{1}{1} = 1$$

   (c) The reciprocal of $\frac{23}{75}$ is $\frac{75}{23}$.

   $$\frac{23}{75} \times \frac{75}{23} = \frac{{}^{1}\cancel{23}}{{}_{1}\cancel{75}} \times \frac{\cancel{75}^{1}}{\cancel{23}_{1}} = \frac{1}{1} = 1$$

7. Division by a number is the same as multiplication by the reciprocal of that number.

   (a) What is $\frac{1}{2}$ divided by $\frac{1}{2}$?

   $$\frac{1}{2} \div \frac{1}{2} = \frac{1}{2} \times \frac{2}{1} = 1$$

   This is otherwise obvious too, because any number divided by itself equals one. Fractions are no exceptions.

   (b) What is 2 divided by $\frac{1}{5}$?

   $$2 \div \frac{1}{5} = \frac{2}{1} \times \frac{5}{1} = 10$$

   This is obvious too, when we think in terms of how many "one fifths" pieces may be obtained from 2 pizzas.

   (c) Divide $\frac{9}{16}$ by $\frac{3}{8}$.

   $$\frac{9}{16} \div \frac{3}{8} = \frac{{}^{3}\cancel{9}}{{}_{2}\cancel{16}} \times \frac{\cancel{8}^{1}}{\cancel{3}_{1}} = \frac{3}{2} = 1\frac{1}{2}$$

8. When mixed numbers are involved in a division, convert them to improper fractions first.

   Divide $6\frac{2}{5}$ by $2\frac{2}{15}$

   $$6\frac{2}{5} \div 2\frac{2}{15} = \frac{32}{5} \div \frac{32}{15} = \frac{\cancel{32}}{\cancel{5}} \times \frac{\cancel{15}^{3}}{\cancel{32}} = \frac{3}{1} = 3$$

☺ Exercise B5.3

1. Multiply the following.

(a) $\frac{1}{2} \times \frac{1}{3}$ (d) $\frac{6}{7} \times \frac{1}{3}$ (g) $\frac{2}{3} \times \frac{7}{5}$ (j) $\frac{2}{3} \times \frac{3}{4} \times \frac{4}{5}$

(b) $\frac{2}{5} \times \frac{2}{3}$ (e) $\frac{3}{10} \times \frac{4}{3}$ (h) $\frac{5}{9} \times \frac{5}{6}$ (k) $\frac{2}{3} \times \frac{4}{5} \times \frac{6}{7}$

(c) $\frac{5}{8} \times \frac{2}{3}$ (f) $\frac{6}{35} \times \frac{5}{6}$ (i) $\frac{7}{8} \times \frac{8}{21}$ (l) $\frac{1}{3} \times \frac{5}{8} \times \frac{9}{11}$

2. Reduce the following to lowest terms during multiplication.

(a) $\frac{3}{7} \times \frac{7}{9}$ (d) $\frac{6}{7} \times \frac{14}{15}$ (g) $\frac{3}{14} \times \frac{7}{15}$ (j) $\frac{1}{3} \times \frac{9}{13} \times \frac{26}{27}$

(b) $\frac{3}{5} \times \frac{5}{3}$ (e) $\frac{9}{10} \times \frac{5}{18}$ (h) $\frac{5}{14} \times \frac{7}{15}$ (k) $\frac{5}{3} \times \frac{6}{7} \times \frac{3}{2}$

(c) $\frac{3}{8} \times \frac{4}{9}$ (f) $\frac{6}{35} \times \frac{14}{9}$ (i) $\frac{7}{8} \times \frac{6}{49}$ (l) $\frac{5}{3} \times \frac{9}{11} \times \frac{33}{45}$

3. Find the following fractions of an hour in minutes.

(a) $\frac{5}{12}$ (b) $\frac{3}{4}$ (c) $\frac{1}{2}$ (d) $\frac{5}{6}$ (e) $\frac{8}{15}$ (f) $\frac{17}{20}$ (g) $\frac{17}{30}$ (h) $\frac{2}{3}$

4. Find the following fractions of a dollar in cents.

(a) $\frac{2}{5}$ (b) $\frac{3}{5}$ (c) $\frac{4}{5}$ (d) $\frac{7}{10}$ (e) $\frac{1}{2}$ (f) $\frac{17}{20}$ (g) $\frac{17}{25}$ (h) $\frac{17}{50}$

5. Determine the following.

(a) A third of a yard in feet. (1 yard = 3 feet)

(b) $\frac{2}{3}$ of a foot in inches.

(c) $\frac{3}{4}$ of a minute in seconds.

(d) $\frac{3}{4}$ of a pound in ounces. (1 pound = 16 ounces)

(e) A tenth of a mile in yards. (1 mile = 1760 yards)

(f) $\frac{5}{8}$ of a mile in yards.

(g) $\frac{4}{7}$ of a class of 35 students.

(h) $\frac{6}{13}$ of a class of 39 students.

6. $\frac{2}{3}$rd of a graduating class of 72 is girls. How many girls are graduating?

7. John played 20 games and won $\frac{4}{5}$th of them. How many games were won?

8. In a 24 hour day, Bill spends $\frac{1}{4}$th of the day in school, $\frac{1}{8}$ th of the day doing his homework, and $\frac{1}{3}$rd of the day sleeping. How many hours in a day does he spend in each of these activities? Are there any hours remaining?

9. Multiply the following.

(a) $7\frac{7}{8} \times 9\frac{1}{7}$ (c) $1\frac{5}{9} \times 1\frac{15}{21}$ (e) $1\frac{1}{3} \times 1\frac{1}{5} \times 1\frac{7}{8}$

(b) $5\frac{3}{7} \times 1\frac{2}{19}$ (d) $2\frac{1}{3} \times 1\frac{13}{14}$ (f) $1\frac{8}{13} \times 2\frac{8}{9} \times 3\frac{3}{4}$

10. Jill bought $5\frac{1}{2}$ pound of butter at \$ $1\frac{3}{4}$ per pound. How much money did she pay for that butter?

11. Determine the reciprocals of the following.

(a) $\frac{1}{3}$	(d) $\frac{7}{8}$	(g) $\frac{20}{13}$
(b) 3	(e) 14	(h) $\frac{25}{33}$
(c) $\frac{2}{5}$	(f) $\frac{11}{16}$	(i) 100

12. Divide the following.

(a) $\frac{3}{7} \div \frac{4}{7}$	(d) $\frac{8}{15} \div \frac{8}{9}$	(g) $\frac{15}{17} \div \frac{35}{51}$	(j) $\frac{2}{3} \div \frac{3}{4}$
(b) $\frac{7}{9} \div \frac{5}{4}$	(e) $\frac{6}{35} \div \frac{3}{7}$	(h) $\frac{7}{8} \div \frac{21}{16}$	(k) $\frac{5}{3} \div \frac{9}{11}$
(c) $\frac{9}{16} \div \frac{5}{8}$	(f) $\frac{9}{10} \div \frac{7}{20}$	(i) $\frac{21}{33} \div \frac{42}{99}$	(l) $\frac{11}{18} \div \frac{2}{5}$

13. If a bird covers $\frac{9}{8}$ feet in $\frac{3}{4}$ of a second, what is its speed in feet per second?

14. Divide the following.

(a) $3\frac{2}{3} \div 3\frac{2}{3}$	(c) $5\frac{5}{7} \div \frac{10}{21}$	(e) $8\frac{1}{4} \div 3\frac{2}{3}$
(b) $3\frac{2}{3} \div \frac{11}{18}$	(d) $5\frac{5}{7} \div 1\frac{11}{14}$	(f) $3\frac{3}{4} \div 3\frac{1}{8}$

15. Mary bought $3\frac{3}{4}$ pound of sugar for $1\frac{2}{3}$ dollars. What is the cost of sugar per pound?

16. Joe drove $61\frac{1}{5}$ miles in $1\frac{1}{2}$ hours. What was his average speed in miles per hour?

17. Practice multiplication and division with fractions from other math books available on the market or by creating exercises of your own.

Answer: **1.** (a) 1/6 (b) 4/15 (c) 10/24 or 5/12 (d) 6/21 or 2/7 (e) 12/30 or 2/5 (f) 30/210 or 1/7 (g) 14/15 (h) 25/54 (i) 56/168 or 1/3 (j) 24/60 or 2/5 (k) 48/105 or 16/35 (l) 45/264 or 15/88 **2.** (a) 1/3 (b) 1 (c) 1/6 (d) 4/5 (e) 1/4 (f) 4/15 (g) 1/10 (h) 1/6 (i) 3/28 (j) 2/9 (k) 15/7 (l) 1 **3.** (a) 25 (b) 45 (c) 30 (d) 50 (e) 32 (f) 51 (g) 34 (h) 40 **4.** (a) 40 (b) 60 (c) 80 (d) 70 (e) 50 (f) 85 (g) 68 (h) 34 **5.** (a) 1 foot (b) 8 inches (c) 45 seconds (d) 12 ounces (e) 176 yards (f) 1100 yards (g) 20 students (h) 18 students **6.** 48 **7.** 16 **8.** 6 hours in school, 3 hours doing homework, and 8 hours sleeping with 7 hours remaining. **9.** (a) 72 (b) 6 (c) 2 2/3 (d) 4 1/2 (e) 3 (f) 17 1/2 **10.** \$9 5/8 **11.** (a) 3 (b) 1/3 (c) 5/2 (d) 8/7 (e) 1/14 (f) 16/11 (g) 13/20 (h) 33/25 (i) 1/100 **12.** (a) 3/4 (b) 28/45 (c) 9/10 (d) 3/5 (e) 2/5 (f) 18/7 (g) 9/7 (h) 2/3 (i) 3/2 (j) 8/9 (k) 55/27 (l) 55/36 **13.** 3/2 ft/sec **14.** (a) 1 (b) 6 (c) 12 (d) 3 1/5 (e) 2 1/4 (f) 1 1/5 **15.** 4/9 dollars per pound **16.** 40 4/5 miles per hour

Lesson B5.4 Mixed Operations

When several fractions are operated upon together by addition and subtraction, we have mixed operations.

1. One may convert all the fractions to like fractions first, and then apply mixed operations to the numerators.

(a) Simplify $\frac{5}{6} - \frac{1}{2} + \frac{2}{3} - \frac{5}{9}$

$$= \frac{15 - 9 + 12 - 10}{18}$$

$$= \frac{8}{18}$$

$$= \frac{4}{9}$$

Alternatively, we may collect together minuends and subtrahends first, and then subtract.

$$= \left(\frac{5}{6} + \frac{2}{3}\right) - \left(\frac{1}{2} + \frac{5}{9}\right)$$

$$= \frac{9}{6} - \frac{19}{18}$$

$$= \frac{8}{18}$$

$$= \frac{4}{9}$$

2. Complex fractions are made up of mixed operations that may be reduced to simple fractions.

(a) Simplify $\dfrac{\frac{5}{7} - \frac{4}{21}}{\frac{11}{21} + \frac{3}{14}}$

This fraction may be written as a mixed operation as follows.

$$\frac{\frac{5}{7} - \frac{4}{21}}{\frac{11}{21} + \frac{3}{14}} = \left(\frac{5}{7} - \frac{4}{21}\right) \div \left(\frac{11}{21} + \frac{3}{14}\right)$$

$$= \left(\frac{15 - 4}{21}\right) \div \left(\frac{22 + 9}{42}\right)$$

$$= \frac{11}{21} \div \frac{31}{42}$$

$$= \frac{11}{21} \times \frac{42}{31}$$

$$= \frac{22}{31}$$

(b) Simplify $\dfrac{\left(\frac{2}{3} \times \frac{9}{16}\right) + \left(\frac{3}{8} \div \frac{15}{32}\right)}{\left(3\frac{1}{3} \div 2\frac{1}{12}\right) - \left(1\frac{1}{9} \times 1\frac{1}{5}\right)}$

Writing it as a mixed operation, we get

$$= \left[\left(\frac{2}{3} \times \frac{9}{16}\right)+\left(\frac{3}{8} \div \frac{15}{32}\right)\right] \div \left[\left(3\frac{1}{3} \div 2\frac{1}{12}\right)-\left(1\frac{1}{9} \times 1\frac{1}{5}\right)\right]$$

$$= \left[\left(\frac{2}{3} \times \frac{9}{16}\right)+\left(\frac{3}{8} \times \frac{32}{15}\right)\right] \div \left[\left(\frac{10}{3} \times \frac{12}{25}\right)-\left(\frac{10}{9} \times \frac{6}{5}\right)\right]$$

$$= \left[\left(\frac{3}{8}\right)+\left(\frac{4}{5}\right)\right] \div \left[\left(\frac{8}{5}\right)-\left(\frac{4}{3}\right)\right]$$

$$= \left[\frac{15+32}{40}\right] \div \left[\frac{24-20}{15}\right]$$

$$= \frac{47}{40} \div \frac{4}{15}$$

$$= \frac{47}{40} \times \frac{15}{4}$$

$$= \frac{141}{32} \quad = \quad 4\frac{13}{32}$$

☺ Exercise B5.4

1. Simplify the following.

(a) $\frac{3}{4} + \frac{5}{6} - \frac{11}{12}$ (d) $\frac{2}{9} - \frac{5}{12} - \frac{1}{4} + \frac{1}{3}$

(b) $\frac{4}{5} - \frac{1}{4} - \frac{3}{15}$ (e) $\frac{5}{14} + \frac{3}{7} - \frac{3}{4} + \frac{4}{21}$

(c) $\frac{4}{15} - \frac{7}{12} + \frac{3}{10}$ (f) $\frac{15}{16} - \frac{3}{24} - \frac{7}{12} + \frac{1}{6} - \frac{3}{8}$

2. Bill, Sam, and Joe won a lottery. Bill's share was $\frac{4}{11}$ of the total. Sam's share was $\frac{10}{33}$ of the total. Rest was John's share. What fraction of the lottery did John get?

3. Simplify the following.

(a) $$\frac{\frac{7}{8} + \frac{11}{12}}{1\frac{1}{16} - \frac{1}{6}}$$

(b) $$\frac{\left(8\frac{2}{5} \div 2\frac{1}{10}\right)-\left(\frac{6}{7} \times 2\frac{11}{12}\right)}{\left(1\frac{1}{4} \times \frac{2}{3}\right)+\left(1\frac{5}{9} \div 2\frac{1}{3}\right)}$$

4. Solve similar problems from books available on the market.

Answer: **1.** (a) 2/3 (b) 7/20 (c) -1/60 (d) -1/9 (e) 19/84 (f) 1/48 **2.** 1/3 **3.** (a) 2 (b) 1

Lesson B5.5 Problem Solving

Problem solving starts with a translation from English to math. To solve, one breaks down the problem into sequential steps. This requires logical thinking.

1. A school athletic field is 120 yards long and 60 yards wide. If the grass on $\frac{2}{3}$ of it had to be reseeded, how many square yards were reseeded?

$$\begin{aligned}
\text{Area of the athletic field} &= 120 \text{ yd.} \times 60 \text{ yd.} \\
&= (120 \times 60) \text{ sq. yd.} \\
\text{Area to be reseeded} &= \tfrac{2}{3} \text{ of the area of the athletic field} \\
&= \tfrac{2}{3} \text{ of } (120 \times 60) \text{ sq. yd.} \\
&= \frac{2}{\cancel{3}} \times \overset{40}{\cancel{120}} \times 60 \\
&= 2 \times 40 \times 60 \\
&= 4800 \text{ square yards}
\end{aligned}$$

2. A family used $29\frac{1}{2}$ square yards of carpet for their living room, $12\frac{1}{3}$ square yards for a bedroom, and $15\frac{1}{4}$ square yards for the hall.

   (a) What was the total amount of carpet used?

   (b) If the carpet cost them $12 a square yard laid, how much did it cost them for the complete job?

<u>Solution</u>:

(a) Total amount of carpet (sq.yd.) = $29\frac{1}{2} + 12\frac{1}{3} + 15\frac{1}{4} = 57\frac{1}{12}$

(b) Cost for the complete job ($) = $57\frac{1}{12} \times 12$ = $685

3. An ancient Bedouin willed one-half of his camels to his oldest son, one-third to his second son, and one-ninth to his youngest son. When the father died, he had 17 camels. The sons could not see how to divide this herd according to their inheritance without killing some of the camels. They called on an uncle to solve this problem for them. The uncle added one of his own camels to the herd, making it 18 camels in all, then gave each son his share of the 18 camels.

oldest son: $\frac{1}{2}$ of 18 = 9 camels

second son: $\frac{1}{3}$ of 18 = 6 camels

youngest son: $\frac{1}{9}$ of 18 = 2 camels   Total = 17 camels

How was it possible for each son to receive more than his share of the 17 camels, and at the same time have the uncle get his camel back?

ANSWER: Note that $\frac{1}{2} + \frac{1}{3} + \frac{1}{9} = \frac{9}{18} + \frac{6}{18} + \frac{2}{18} = \frac{17}{18}$

The sons were only to receive 17/18 of the 17 camels. They actually got all 17 camels, which was slightly more than their proper shares, in order that no camel would have to be cut up.

☺ Exercise B5.5

1. Find the area reseeded in the first problem above, if
   (a) The length of the field was 100 yards
   (b) The width of the field was 80 yards
   (c) $\frac{4}{5}$ of the field was reseeded

2. Solve the second problem above, if the carpet used was $36\frac{2}{3}$, $23\frac{3}{4}$, and $21\frac{5}{6}$ square yards respectively, and the cost was $8 a square yard laid.

3. How many square yards of carpet will be needed to carpet a rectangular room that is $7\frac{1}{2}$ yards long and $4\frac{2}{3}$ yards wide?

4. A board $96\frac{1}{2}$ inches long is cut into three pieces of equal length. If $\frac{1}{4}$ inches is lost each time the board is sawed, how long is each of the three finished pieces?

5. Solve word problems containing fractions from books available on the market.

Answer: **1.** (a) 4000 sq. yd. (b) 6400 sq. yd. (c) 5760 sq. yd. **2.** (a) 82 1/4 sq. yd. (b) $ 658
3. 35 sq. yd. **4.** 32 in. because 1/2 in. of 96 1/2 in. was lost in the two cuts made.

SUMMARY

Like fractions are added by adding the numerators. *Like fractions* are subtracted by subtracting the numerators. The denominator remains the same. To add or subtract *unlike fractions*, one must convert them to *like fractions* first.

To convert unlike to like fractions, we first calculate the LCM (least common multiple) of all the unlike denominators. Then we calculate the equivalent fractions for unlike fractions with the LCM as the new denominator.

To multiply fractions, we simply multiply the numerators together to get the numerator of the product, and multiply the denominators together to get the denominator of the product. To divide by a fraction, we simply multiply by its reciprocal.

In general practice, a fraction in the final answer is expressed in its lowest terms. The lowest terms are obtained by taking all the common factors out of the numerator and the denominator.

A "division" notation is not the only notation possible to express fractions. Another way is to extend the place value notation to account for fractions. That notation is covered under the milestone on DECIMAL NUMBERS.

DIAGNOSTIC TEST

1. What is the least common multiple of 9, 14, and 21?
2. Add $\frac{5}{6}$, $\frac{9}{14}$ and $\frac{11}{21}$.
3. John, Bill, and Mike shared a whole pizza. John ate $\frac{1}{4}$, Bill ate $\frac{1}{3}$, and Mike ate $\frac{5}{12}$ of the pizza. How much of the pizza was left?
4. Subtract $2\frac{5}{8}$ from $3\frac{7}{12}$.
5. Multiply $\frac{8}{27}$ by $\frac{15}{16}$ by first reducing the product to simplest terms during multiplication.
6. A person must pay $\frac{1}{6}$th of his income in taxes. If the annual income of Joe is $36,000, how much should he pay in taxes?
7. How many seconds are $\frac{3}{4}$ of a minute?
8. Multiply $1\frac{3}{5}$ by $1\frac{9}{16}$.
9. Divide $\frac{9}{16}$ by $\frac{3}{8}$.
10. Divide $6\frac{2}{5}$ by $2\frac{2}{15}$.
11. Express "4 inches" as a fraction of a foot.
12. Simplify $\frac{5}{6} - \frac{1}{2} + \frac{2}{3} - \frac{5}{9}$
13. Simplify $\dfrac{\frac{5}{7} - \frac{4}{21}}{\frac{11}{21} + \frac{3}{14}}$

Answer: **1.** 126 **2.** 2 **3.** None **4.** 23/24 **5.** 5/18 **6.** $6000 **7.** 45 **8.** 2 1/2 **9.** 3/2 **10.** 3 **11.** 1/3 ft **12.** 4/9 **13.** 22/31

<u>GLOSSARY</u>

[For additional words refer to the glossaries at the end of earlier Milestones]

Common multiple When a number is added to itself repeatedly, we obtain its **multiples**. A multiple is divisible by the number. A **common multiple** of two numbers is divisible by both numbers.

LCM An **LCM (Least Common Multiple)** of two or more numbers is the smallest number that is divisible by those numbers.

Least common multiple See **LCM**

Ratio A **ratio** is the relation between two quantities expressed as the quotient of one divided by the other. For example, the ratio of 16 to 8 is 16/8 or 2, which means "16 is <u>twice</u> of 8. The ratio of 8 to 16 is written 8:16 or 8/16 or ½, and it means "half."

Reciprocal The **reciprocal** of a number is obtained by switching the numerator and the denominator. For example, the reciprocal of 3/4 is 4/3, and the reciprocal of 2 is 1/2. The product of a number and its reciprocal is always 1.